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Singular Yukawa and gauge couplings in $d = 4$ Heterotic String Vacua[1]

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ABSTRACT

In this paper we discuss the singularities in the Yukawa and gauge couplings of $N = 1$ compactifications of the $SO(32)$ heterotic string in four space-time dimensions. Such singularities can arise from the strong coupling dynamics of a confined non-perturbative gauge group.

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1 Introduction

The recent advances in string theory have led to a much better understanding of its non-perturbative properties. One particular aspect of these developments concerns couplings in the low energy effective action, which are singular functions on the moduli space of a given family of perturbative string vacua. Whenever such couplings obey a non-renormalization theorem, the singularity cannot be smoothed out by (perturbative) quantum corrections but must have a physical origin. An example of this situation is found in $K3$ compactifications of the heterotic string which have six space-time dimensions ($d = 6$) and 8 supercharges. For consistency, such vacua necessarily have non-trivial instantons embedded in the gauge group $E_8 \times E_8$ or $SO(32)$. Associated with these instantons is a (quaternionic) moduli space which parametrizes the size, location, and embedding of the instantons into the gauge group. This moduli space has singularities at points where instantons shrink to zero size. It has been convincingly argued that the physical origin of these singularities is either a set of gauge bosons becoming massless or an entire string becoming tensionless [1-8]. Both of these effects are invisible in string perturbation theory in that they occur in regions of the moduli space where the perturbation theory breaks down.

A similar situation is found in type II vacua compactified on Calabi–Yau threefolds which have $N = 2$ supersymmetry in $d = 4$. In this case the gauge couplings generically have logarithmic singularities on the moduli space of the scalar fields in the $N = 2$ vector multiplets. Strominger [9] suggested that these singularities are caused by non-perturbative charged states in the type II string spectrum which become massless at the singularity. In perturbative string theory all such states are heavy and hence are integrated out of the effective action. Their only remnant visible in perturbation theory are the moduli dependent (and singular) gauge couplings. However, whenever some of the non-perturbative states become massless it is no longer legitimate to integrate them out, and thus the singularity signals additional light states and the break down of an effective low energy theory which does not properly include all the light degrees of freedom. The non-perturbative states arise in $N = 2$ hypermultiplets and carry $U(1)$ charge of Ramond–Ramond vector bosons. As a consequence, they do not have the canonical couplings to the dilaton [10] but their mass only depends on the scalar fields of the $N = 2$ vector multiplets. Therefore, the corrections to the gauge couplings – even though being non-perturbative – appear without any dilaton dependence and thus 'compete' with tree level effects.

In $N = 1$ heterotic string vacua compactified on a Calabi–Yau manifold or, more generally, on $(0, 2)$ superconformal field theories (SCFTs) the situation is more complicated. In this case one typically has power-like singularities of the tree level Yukawa couplings in addition to logarithmic singularities of the gauge couplings at one-loop. The power-like tree level singularities of the Yukawa couplings cannot be explained by states becoming massless. Instead it was suggested by Kachru, Seiberg and Silverstein [11] that at least some of the singularities are caused by strong coupling dynamics of an asymptotically free non-pertur-

bative gauge group. As we already stated, non-perturbative gauge groups do arise in $d = 6$ vacua of the heterotic string. For example, $Sp(2k)$ appears when k small $SO(32)$ instantons shrink on the same smooth point in the $K3$ [1] while $U(2k)$ is the non-perturbative gauge group for k small instantons (without vector structure) shrinking on a singular point of $K3$ [12-14,5-8]. It has been shown that for a specific class of Calabi–Yau compactifications the non-perturbative gauge group 'descends down' to $d = 4$ and can be responsible for the singular couplings of the effective action [11].

In this paper we expand on the mechanism of ref. [11] in two respects. We show that generically there can be states in the string spectrum which become massless at the singularity and thus are responsible for the additional logarithmic singularity of the gauge couplings at one-loop (section 2). Furthermore, ref. [11] concentrated on a one-dimensional moduli space and a non-perturbative gauge group $SU(2)$. The generalization to higher dimensional moduli spaces with a more complicated structure of the singularities and non-perturbative gauge groups $Sp(2k)$ and $U(2k)$ are presented in sections 3 and 4 respectively.

A summary of our results appeared previously in ref. [15].

2 Heterotic vacua with non-perturbative gauge group $SU(2)$

Perturbative $d = 4$, $N = 1$ heterotic vacua are characterized by a $c = 9$ SCFT with $(0, 2)$ worldsheet supersymmetry and the choice of a $\bar{c} = 22$ vector bundle.[3] The space-time spectrum features the gravitational multiplet, non-Abelian vector multiplets, charged chiral matter multiplets $Q^I, \hat{Q}^{\hat{I}}$, and gauge neutral chiral moduli multiplets M^i. The couplings of these multiplets are described by an effective Lagrangian which is constrained by $N = 1$ supersymmetry to only depend on three arbitrary functions: the real Kähler potential K, the holomorphic superpotential W, and the holomorphic gauge kinetic function f. Due to their holomorphicity the latter two obey a non-renormalization theorem [16]. The superpotential W receives no perturbative corrections, and one only has

$$W = W^{(0)} + W^{(\mathrm{NP})} , \tag{1}$$

where $W^{(0)}$ denotes the tree level contribution while $W^{(\mathrm{NP})}$ summarizes the non-perturbative corrections. $W^{(0)}$ contains mass terms and Yukawa couplings both of which are generically moduli dependent. $\hat{Q}^{\hat{I}}$ are massive charged multiplets of the string vacuum while Q^I denotes the massless multiplets. The massive modes $\hat{Q}^{\hat{I}}$ are commonly integrated out of the effective Lagrangian since their typical mass is of order of the Planck scale M_{Pl}. However, since their masses can be moduli dependent, they might become light in special regions of the moduli

[3] An important subset of such vacua are the geometrical Calabi–Yau compactifications which have $(2, 2)$ worldsheet supersymmetry.

space, and therefore we choose to keep them in the effective theory. Thus a generic tree level superpotential is given by

$$W^{(0)} = m_{\hat{I}}(M^i)\,\hat{Q}^{\hat{I}}\hat{Q}^{\hat{I}} + Y_{IJK}(M^i)\,Q^I Q^J Q^K + \dots \ , \tag{2}$$

where all gauge quantum numbers of Q^I and $\hat{Q}^{\hat{I}}$ are suppressed.

The real part of the gauge kinetic function f determines the inverse gauge coupling according to $g^{-2} = \mathrm{Re}f$. The holomorphic f receives no perturbative corrections beyond one-loop, and one has

$$f = f^{(0)} + f^{(1)} + f^{(\mathrm{NP})} \ , \tag{3}$$

where $f^{(1)}$ is the one-loop correction.

In most (if not all) heterotic string vacua W and f are singular functions on the moduli space. (For a recent discussion see ref. [17].) For example, in a heterotic vacuum obtained as a Calabi–Yau compactification on a quintic hypersurface in $\mathbf{CP^4}$ with defining polynomial $p = \sum_{\alpha=1}^{5} X_\alpha^5 - 5\psi X_1 X_2 X_3 X_4 X_5$ one finds [18, 19]

$$Y \sim M^{-1}, \quad f \sim \log M \ , \tag{4}$$

where $M \equiv 1 - \psi^5$. It has been suggested in ref. [11] that such singularities are consequences of a strongly coupled non-perturbative gauge group. Such non-perturbative gauge groups are known to arise in the $d = 6$ $SO(32)$ heterotic string at points where instantons shrink to zero size. The moduli space of k $SO(32)$ instantons at smooth points of $K3$ is isomorphic to the Higgs branch of an $Sp(2k)$[4] gauge theory with 32 half-hypermultiplets in the fundamental ($\Box$) $2k$-dimensional representation, a traceless antisymmetric tensor ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$) in the $k(2k-1)-1$-dimensional representation, and a singlet [1]. The singularities of this moduli space precisely occur where some (or all) of the $Sp(2k)$ gauge bosons become massless.[5] Upon toroidally compactifying the above theories one obtains $N = 2$ string vacua in $d = 4$. String vacua with $N = 1$ supersymmetry arise when one compactifies not on $K3 \times T^2$ but on a Calabi–Yau threefold. However, there is a particular class of threefolds – $K3$ fibrations – which is closely related to the six-dimensional heterotic vacua. For such vacua a $K3$ is fibred over a $\mathbf{P^1}$ base, and if the base is large the adiabatic argument applies [22], and the singularities of the $K3$ fibres are inherited from the corresponding six-dimensional vacuum [11]. In this case the gauge group G of the four-dimensional string vacuum is a product of the perturbative gauge group G_{pert} and the non-perturbative gauge group G_{NP}

$$G = G_{\mathrm{pert}} \times G_{\mathrm{NP}} \ , \tag{5}$$

where G_{NP} is a subgroup of $Sp(2k)$.

[4] By $Sp(2k)$ we mean the rank k symplectic group whose fundamental representation has dimension $2k$.

[5] Singularities in the $E_8 \times E_8$ heterotic string and chirality changing phase transitions have recently been discussed in refs. [20, 21].

A specific class of $K3$ fibrations are the quintic hypersurfaces defined in weighted projective space $\mathbf{WP^4}_{1,1,2k_1,2k_2,2k_3}$ [23, 24]. For compactifications of the $SO(32)$ heterotic string on $(0,2)$ deformations of such Calabi–Yau spaces the non-perturbative spectrum is computed in ref. [11] for the case of a single small instanton at a smooth point in the $K3$ fibre.[6] It is found that the non-perturbative gauge group in the four-dimensional vacuum is given by $G_{\rm NP} = Sp(2) \cong SU(2)$, and out of the 32 half-hypermultiplets in $d = 6$ only four $SU(2)$ doublets in chiral $N = 1$ supermultiplets (which we denote by $q_i^\alpha, i = 1, \ldots, 4, \alpha = 1, 2$) survive in $d = 4$. This resulting gauge theory is asymptotically free ($b_{SU(2)} > 0$) and thus becomes strongly coupled below its characteristic scale $\Lambda_{SU(2)}$. It has the additional property that

$$c := T(\mathbf{ad}) - \sum_r n_r T(r) = 2 - 4 \cdot \frac{1}{2} = 0 \ , \tag{6}$$

where $T(r)$ is the index in the representation r, $T(\mathbf{ad})$ is the index in the adjoint representation, and n_r counts the number of chiral multiplets in representation r. (In this notation the one-loop coefficient of the β-function is given by $b = 3T(\mathbf{ad}) - \sum_r n_r T(r)$.)

The coefficient c also appears in the anomaly equation of the R-symmetry. An anomaly free R-symmetry imposes $\sum_r n_r T(r)\, R_r = -c$, where R_r is the R-charge of a superfield in representation r. Therefore, in gauge theories with $c = 0$ one can always choose $R = 0$ for all superfields, and, as a consequence, no non-perturbative superpotential $W^{(\rm NP)}$ (which necessarily carries $R = 2$) can be generated by the strong coupling dynamics [25, 26]. This conclusion is believed to hold irrespective of the precise form of the tree level superpotential.

The $SU(2)$ gauge theory under consideration confines below $\Lambda_{SU(2)}$, and the surviving degrees of freedom are the gauge singlets

$$M_{ij} := q_i^\alpha \epsilon_{\alpha\beta} q_j^\beta \ . \tag{7}$$

M_{ij} is antisymmetric, and due to Bose statistics of the quark superfields it obeys the constraint ${\rm Pf} M := \frac{1}{8}\epsilon^{ijkl} M_{ij} M_{kl} = 0$; thus there are five physical degrees of freedom in the effective theory. Quantum mechanically the constraint is modified and reads [27]

$${\rm Pf} M = \Lambda^4_{SU(2)} \ . \tag{8}$$

Non-renormalizable interactions typically induce mass terms as well as higher dimensional couplings for some of the M_{ij}, and thus not all scalar degrees of freedom are moduli of the low energy theory. Altogether the superpotential is given by

$$\begin{aligned} W = m_{\hat I}(M)\, \hat Q^{\hat I} \hat Q^{\hat I} + Y_{IJK}(M)\, Q^I Q^J Q^K + \lambda({\rm Pf}M - \Lambda^4) + \sum_{i<j} m_{ij} M_{ij}^2 \\ + \text{ non-renormalizable terms } + \text{stringy non-perturbative terms} \ , \end{aligned} \tag{9}$$

[6] It is important to consider a $(0,2)$ deformation since on the $(2,2)$ locus the spin connection is embedded in the gauge connection, and a small instanton necessarily has to shrink on a $K3$ singularity. This situation is not fully understood at present. (We thank P. Aspinwall and K. Intriligator for a useful correspondence on this point.)

where λ is a Lagrange multiplier and we have choosen $M_{\rm Pl}=1$. Note that even though no non-perturbative superpotential is generated by the strongly coupled gauge theory, there are 'stringy' non-perturbative corrections of order $\mathcal{O}(e^{-g_{\rm string}^{-2}})$ where $g_{\rm string}$ is the string coupling constant which is related to vacuum expectation value of the dilaton.

The supersymmetric minima are determined by the solution of $\partial W=0$, where the derivative has to be taken with respect to all fields in the theory. The matter fields $Q^I, \hat{Q}^{\hat{I}}$ of a string vacuum are always choosen to obey $\langle Q^I\rangle = \langle \hat{Q}^{\hat{I}}\rangle = 0$ which leaves us with

$$\frac{\partial W}{\partial M_{ij}} = 0 \quad \to \quad \tfrac{1}{2}\lambda\ \epsilon^{ijkl} M_{kl} + 2 m_{ij} M_{ij} = 0\ , \tag{10}$$

$$\frac{\partial W}{\partial \lambda} = 0 \quad \to \quad \epsilon^{ijkl} M_{ij} M_{kl} = 8\ \Lambda^4 \tag{11}$$

as nontrivial conditions. (We neglect further non-renormalizable interactions, since they are suppressed by $M_{\rm Pl}$. In addition, $g_{\rm string}$ is taken to be weak so that the stringy non-perturbative corrections can also be ignored.)

To further simplify our analysis we only consider solutions of eqs. (10), (11) that satisfy $\langle W\rangle = 0$. This is partly motivated by the fact that $\langle W\rangle \neq 0$ induces a cosmological constant in supergravity and partly by the considerable simplification of the vacuum solutions. Inserting eqs. (10), (11) into eq. (9) and using $\langle Q^I\rangle = \langle \hat{Q}^{\hat{I}}\rangle = 0$ we arrive at

$$\langle W\rangle = \sum_{i<j} m_{ij} \langle M_{ij}\rangle^2 = -\langle\lambda\rangle \Lambda^4\ . \tag{12}$$

Thus $\langle W\rangle = 0$ demands $\langle\lambda\rangle = 0$. This class of solutions has an immediate other consequence. From eq. (10) we learn that each of the M_{ij} whose mass term does not vanish is set to zero. Eq. (11) then implies that at least two masses have to vanish, and they have to be such that $m_{ij} = m_{kl} = 0$ with i,j,k,l all different. Thus, in our approximation the dimension of the moduli space is given by the number of vanishing mass terms minus one.[7]

Ref. [11] considered the choice $m_{12} = m_{34} = 0$ which implies a one-dimensional moduli space and

$$M_{13} = M_{14} = M_{23} = M_{24} = 0\ , \qquad M_{12} M_{34} = \Lambda^4\ , \tag{13}$$

via eqs. (10), (11). (We suppress the brackets to denote VEV's henceforth.)

In string perturbation theory both the Yukawa couplings $Y_{IJK}(M)$ and the masses $m_{\hat{I}}(M)$ are given as power series expansion in the moduli. However, the strong coupling effects which are responsible for generating the non-perturbative constraint ${\rm Pf} M = \Lambda^4_{SU(2)}$ remove the origin of the moduli space and render the perturbative expansion of the Yukawa couplings singular [11]. For example,

$$Y_{IJK}(M) \sim M_{12} + M_{34} + \ldots \tag{14}$$

[7] Strictly speaking the condition for a given M_{ij} to be a modulus involves higher dimensional terms which we have neglected in our analysis.

produces a singularity

$$Y_{IJK}(M) \sim \frac{\Lambda^4}{M_{34}} + \dots \quad \text{as } M_{34} \to 0 \ . \tag{15}$$

(Higher dimensional terms in (14) are Planck suppressed but do produce additional singular terms in eq. (15).)

The $SU(2)$ scale $\Lambda_{SU(2)}$ depends on the gauge coupling $g_{SU(2)}$ via $\Lambda_{SU(2)} \sim \exp(-\frac{8\pi^2}{b g^2_{SU(2)}})$. It was shown in refs. [28, 29] that in $d = 6$ the gauge couplings of the non-perturbative gauge bosons do not depend on $g_{\rm string}$ or, in other words, do not have the canonical couplings to the dilaton. It can be easily seen that the same properties hold in $d = 4$ and hence the Yukawa couplings in eq. (15) do not depend on $g_{\rm string}$ either. Thus, exactly as for the singularities of the $N = 2$ vacua, the non-perturbative effect which generates the singularity in eq. (15) does not have the standard dilaton or $g_{\rm string}$ dependence but instead competes with tree level couplings. (Of course, this is a necessary requirement for a consistent explanation of the singularities in the Yukawa couplings.)

The mass terms $m_{\hat I}$ either become large or small at the singular points in the moduli space. We already learned from the $N = 2$ case [9] that light matter fields $\hat{Q}^{\hat I}$ with $m_{\hat I}(M) \sim M_{34}$ which are charged under $G_{\rm pert}$ produce a singularity in the associated (perturbative) gauge coupling at one-loop; more precisely they contribute a correction

$$g^{-2}_{\rm pert} = \sum_{\hat I} \frac{b_{\hat I}}{16\pi^2} \log|m_{\hat I}|^2 + \dots \ . \tag{16}$$

The coefficient of the singularity is set by the multiplicity of the light modes and their gauge quantum numbers. Unfortunately, in string theory the coefficient of the logarithmic singularity is so far only known for $(2,2)$ vacua [19, 30], and hence a more detailed comparison with the $(0,2)$ models considered in ref. [11] cannot be presented. Conversely, it is not known at present how to repeat the analysis of ref. [11] for $(2,2)$ vacua, since the non-perturbative physics in the corresponding $d = 6$ vacua is not completely understood. Nevertheless it is interesting to display the coefficients of the singularity in $(2,2)$ vacua.[8] One finds for all $(2,2)$ vacua of the $SO(32)$ heterotic string, where $G_{\rm pert} = SO(26) \times U(1)$, the relation

$$16\pi^2 (g^{-2}_{SO(26)} - \tfrac{1}{6} g^{-2}_{U(1)}) = -16 F_1 \ , \tag{17}$$

while for $(2,2)$ vacua of the $E_8 \times E_8$ heterotic string ($G_{\rm pert} = E_8 \times E_6$) one has

$$16\pi^2 (g^{-2}_{E_8} - g^{-2}_{E_6}) = -12 F_1 \ . \tag{18}$$

F_1 is the topological index defined in ref. [19] which for the quintic hypersurface in $\mathbf{CP^4}$ is given by

$$F_1 = -\frac{1}{12} \log M\bar{M} + \dots \ . \tag{19}$$

[8] This was worked out jointly with V. Kaplunovsky.

One can check that eqs. (16)–(19) are not easy to satisfy. Let us first consider a pair of light states in the fundamental representation of $SO(26)$ with $U(1)$ charges $\pm q$. Inserting eq. (16) into (17) one obtains

$$(-2+\frac{52}{6}\,q^2)\ \log|m|^2=-16F_1=\frac{4}{3}\log|M|^2\ , \tag{20}$$

where the last equation used (19). Already for $q=1$ this implies the relation $m=M^{\frac{1}{5}}$. However, the mass of a state given by a fractional power of M is not sensible within our framework. Similarly, there is no sensible solution for fields in two-index tensor or spinor representations of $SO(26)$. Thus we conclude that at the singularity there can be no light states which are charged under $SO(26)$. For a pair of $SO(26)$ singlets with $U(1)$ charges $\pm q$ one finds instead

$$\frac{1}{3}\,q^2\log|m|^2=-16F_1=\frac{4}{3}\log|M|^2\ . \tag{21}$$

This relation can be satisfied by one pair with mass $m=M$ and $U(1)$ charges ± 2 or four pairs with mass $m=M$ and $U(1)$ charges ± 1. (If $m=M^2$ ($m=M^4$) one can also have two (one) pairs with $U(1)$ charges ± 1.) The case of $E_8\times E_6$ was already discussed in ref. [30], where it was shown that no sensible solution exists at all. The difficulty in satisfying eqs. (16)–(19) might indicate that in $(2,2)$ vacua a different mechanism is responsible for the singularities in the gauge couplings as well as in the Yukawa couplings.[9]

So far we considered a one-dimensional moduli space with $m_{12}=m_{34}=0$. By choosing additional mass terms to be zero, one can obtain higher dimensional moduli spaces. However, the singularity of the Yukawa couplings remains essentially unchanged in that it is a smooth function of the additional moduli. This follows immediately from the quantum constraint (11) which is only quadratic in the fields. In order to encounter a more complicated singular structure the quantum constraint has to involve higher powers in the fields. This precisely occurs in $Sp(2k)$ gauge theories which we turn to in the following section.

3 Heterotic vacua with non-perturbative gauge group $Sp(2k)$

In most Calabi–Yau vacua the singularity of the Yukawa couplings has a more complicated structure than just a simple pole. In particular, one observes generically a singular locus with more than one component where the different components can intersect in various ways. Such a behaviour is reproduced by k small instantons located at the same (smooth) point in moduli space or, in other words, by $G_{\rm NP}=Sp(2k)$. For this case the analysis of ref. [11] can

[9]We checked that this conclusion also holds for the singularities in the $K3$ fibre of the two-parameter models of ref. [23] and we suspect that it is valid in general, since the coefficient of the conifold singularities seems to be universal.

be repeated without major modifications, since it does not depend on the non-perturbative gauge group. Thus, out of the 32 half-hypermultiplets in the fundamental ($\square$) representation of the six-dimensional vacua again four chiral $\square$ multiplets ($q_i^\alpha, i = 1, \ldots, 4, \alpha = 1, \ldots, 2k$) remain in $d = 4$. In addition, the gauge bosons as well as the antisymmetric tensor ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) are constants on the $\mathbf{P}^1$ base of the $K3$ fibration and hence they also survive as chiral multiplets. As before, the resulting spectrum in $d = 4$ is an asymptotically free gauge theory with the additional property that $c_{Sp(2k)} = (k+1) - 4 \cdot \frac{1}{2} - (k-1) = 0$ for all values of k. Thus, the R-charge of all fields can be choosen to vanish and no non-perturbative superpotential can be generated by the strongly coupled $Sp(2k)$ gauge theory. Fortunately, this gauge theory has been analysed in some detail in ref. [31] (see also [32, 33]). The physical degrees of freedom below $\Lambda_{Sp(2k)}$ are found to be

$$
\begin{aligned}
M_{ij}^l \;&:=\; q_i \cdot A^l \cdot q_j \;=\; q_i^\alpha A_\alpha^{\;\beta_1} A_{\beta_1}^{\;\beta_2} \cdots A_{\beta_{l-1}}^{\;\gamma} q_{j\gamma}, \qquad i,j = 1, \ldots, 4, \quad l = 0, \ldots, k-1 \;, \\
T_r \;&:=\; \frac{1}{4r} \mathrm{Tr} A^r \;=\; \frac{1}{4r} A_\alpha^{\;\beta_1} A_{\beta_1}^{\;\beta_2} \cdots A_{\beta_{r-1}}^{\;\alpha}, \qquad r = 2, \ldots, k \;,
\end{aligned}
\tag{22}
$$

where A denotes the antisymmetric tensor, and colour indices are lowered using the $Sp(2k)$ invariant J-tensor, $A_\alpha^{\;\beta} = J_{\alpha\gamma} A^{\gamma\beta}$. Exactly as for the $SU(2)$ gauge theory the M_{ij}^l and T_r are not all independent but related by constraint equations. These constraints can be obtained from the confining superpotential $W_{F=6}$ of the theory with six $\square$ fields ($F = 6$) by integrating out two q's [31]. $W_{F=6}$ is uniquely determined by symmetry arguments and the requirement that the equations of motion reproduce the classical constraints [25, 31]. One obtains

$$
W_{F=6} = \frac{1}{\Lambda^{2k+1}} \sum_{l,m,n,\{i_r\}} c_{lmn,i_2 \ldots i_k} \prod_{r=2}^{k} (T_r)^{i_r} M^l M^m M^n \;, \tag{23}
$$

where the sum runs over all integers

$$
\begin{aligned}
0 \leq l, m, n \leq k-1 \;, \quad i_r \geq 0 \;, \\
\text{with} \quad l + m + n + \sum_{r=2}^{k} r\, i_r = 2(k-1) \;,
\end{aligned}
\tag{24}
$$

and the flavour indices of the three M's in eq. (23) are contracted with an 6-index epsilon tensor. Some of the coefficients in (23) may vanish; for $k \leq 4$ they are calculated in [31]. Adding a mass term for two of the quarks (e.g., $W_{mass} = m M_{56}^0$) and integrating out the massive modes, one obtains k constraints on the $F = 4$ moduli space. These constraints are in one-to-one correspondence with algebraic relations which already hold at the classical level among the fields. Only one of these relations receives quantum corrections; it has the form

$$
\sum_{l,m,\{i_r\}} \tilde{c}_{lm,i_2 \ldots i_k} \prod_{r=2}^{k} (T_r)^{i_r} M^l M^m = \Lambda^{2k+2} \;, \tag{25}
$$

with

$$
l + m + \sum_{r=2}^{k} r\, i_r = 2(k-1) \;. \tag{26}
$$

The other $k-1$ constraints are identical to their classical counterparts. They are similar to eq. (25) but the right hand side vanishes.[10] In addition the condition (26) is modified to $l+m+\sum_{r=2}^{k} r\, i_r = 2(k-1)-p$, with $p=1,\ldots,k-1$.

For simplicity we focus on $k=2$, or $Sp(4)$, henceforth. For this case one has [31]

$$
\begin{aligned}
T_2\, \mathrm{Pf} M^0 + \tfrac{1}{2}\mathrm{Pf} M^1 &= 2\Lambda^6 , \\
\epsilon^{ijkl} M^0_{ij} M^1_{kl} &= 0 ,
\end{aligned}
\tag{27}
$$

where classically the expression on the left hand side of the first equation vanishes. Hence, there are $2\cdot 6+1-2=11$ physical degrees of freedom in the effective theory. Exactly as in the previous $SU(2)$ example these states can be massive, and the superpotential has the generic form

$$
\begin{aligned}
W= \; & m_{\hat{I}}(M,T)\, \hat{Q}^{\hat{I}} \hat{Q}^{\hat{I}} + Y_{IJK}(M,T)\, Q^I Q^J Q^K \\
& + \lambda\, (T_2\, \mathrm{Pf} M^0 + \tfrac{1}{2}\mathrm{Pf} M^1 - 2\Lambda^6) + \mu\, \tfrac{1}{4}\epsilon^{ijkl} M^0_{ij} M^1_{kl} \\
& + \sum_{i<j} \Big(m^0_{ij} (M^0_{ij})^2 + m^1_{ij} (M^1_{ij})^2 \Big) + m T_2^2 + \ldots ,
\end{aligned}
\tag{28}
$$

where λ, μ are the Lagrange multipliers incorporating the constraints.

Varying with respect to $M^0_{ij},\ M^1_{ij},\ T_2,\ \lambda$ and μ we obtain 15 equations of motion:

$$\tfrac{1}{2}\lambda\, T_2\, \epsilon^{ijkl} M^0_{kl} + \tfrac{1}{2}\mu\, \epsilon^{ijkl} M^1_{kl} + 2 m^0_{ij} M^0_{ij} = 0 \quad \forall\, i,j , \tag{29}$$

$$\tfrac{1}{4}\lambda\, \epsilon^{ijkl} M^1_{kl} + \tfrac{1}{2}\mu\, \epsilon^{ijkl} M^0_{kl} + 2 m^1_{ij} M^1_{ij} = 0 \quad \forall\, i,j , \tag{30}$$

$$\tfrac{1}{8}\lambda\, \epsilon^{ijkl} M^0_{ij} M^0_{kl} + 2 m T_2 = 0 , \tag{31}$$

$$T_2\, \epsilon^{ijkl} M^0_{ij} M^0_{kl} + \tfrac{1}{2}\epsilon^{ijkl} M^1_{ij} M^1_{kl} = 16\, \Lambda^6 , \tag{32}$$

$$\epsilon^{ijkl} M^0_{ij} M^1_{kl} = 0 . \tag{33}$$

As before, we only consider solutions for which $\langle W \rangle = 0$ holds. With this additional constraint, eqs. (29)–(33) imply $\lambda = \mu = 0$. This can be seen by solving eq. (29) for $\mu\, \epsilon^{ijkl} M^1_{kl}$ respectively eq. (30) for $\lambda\, \epsilon^{ijkl} M^1_{kl}$ and inserting the result into (33) multiplied by μ respectively (32) multiplied by λ. This yields

$$-\lambda\, T_2\, \epsilon^{ijkl} M^0_{ij} M^0_{kl} - 8 \sum_{i<j} m^0_{ij} (M^0_{ij})^2 = 0 , \tag{34}$$

$$\lambda\, T_2\, \epsilon^{ijkl} M^0_{ij} M^0_{kl} - \mu\, \epsilon^{ijkl} M^1_{ij} M^0_{kl} - 8 \sum_{i<j} m^1_{ij} (M^1_{ij})^2 = 16 \lambda \Lambda^6 , \tag{35}$$

which together with (33) leads to

$$\sum_{i<j} \Big(m^0_{ij} (M^0_{ij})^2 + m^1_{ij} (M^1_{ij})^2 \Big) = -2\lambda\Lambda^6 . \tag{36}$$

[10] These results can also be understood from symmetry considerations [31].

Inserting eqs. (36), (32), and (33) into (28) results in

$$\langle W\rangle = mT_2^2 - 2\lambda\Lambda^6 = 0 \ . \tag{37}$$

All solutions of eqs. (29)–(33), (37) have $\lambda = 0$. In addition, for $\lambda = 0$ eqs. (29) and (30) imply the following six equations:

$$M^0_{kl}(\mu^2 - 4m^1_{ij}m^0_{kl}) = 0 \quad \forall\, i,j,k,l \text{ with } \epsilon^{ijkl} = 1 \ . \tag{38}$$

For generic masses[11] at most one of the M^0_{ij} can be different from zero (otherwise μ would be overdetermined). The case of one non-vanishing M^0_{ij} is not allowed as a consequence of eqs. (29) and (33), while the case of all M^0_{ij} vanishing is not allowed as a consequence of eqs. (29) and (32). Thus eq. (38) has no solution for generic non-vanishing masses. This implies that at least two of the mass terms have to vanish which in turn necessarily sets $\mu = 0$.

$\lambda = \mu = 0$ considerably simplifies the equations of motion. From eqs. (29), (30), (31) we learn that each field with non-vanishing mass term is set to zero. The fields with no mass term are only constrained by eqs. (32), (33). Therefore the dimension of the moduli space is given in general by the number of vanishing mass terms minus two. (However, it is possible that the second constraint is trivially satisfied. In this case the dimension of the moduli space is given by the number of vanishing mass terms minus one.)

The following examples show the different singularity structures of the moduli space appearing for different combinations of mass terms. First, consider the case $m^0_{12} = m^0_{34} = m = 0$ which results in a two-dimensional moduli space. Eqs. (29)–(33) (together with $\lambda = \mu = 0$) lead to

$$\begin{aligned} M^0_{13} = M^0_{14} = M^0_{23} = M^0_{24} &= 0 = M^1_{ij} \quad \forall i,j \ , \\ T_2\, M^0_{12}\, M^0_{34} &= 2\Lambda^6 \ . \end{aligned} \tag{39}$$

A Yukawa coupling of the form

$$Y_{IJK} \sim T_2 + M^0_{12} + M^0_{34} + \ldots \tag{40}$$

now produces a singularity

$$Y_{IJK}(M) \sim \frac{2\Lambda^6}{M^0_{12}M^0_{34}} + \ldots \tag{41}$$

as $M^0_{12} \to 0$ or $M^0_{34} \to 0$. This is an example for a Yukawa coupling which depends on two intersecting singular lines.

As a second example, consider the case $m^1_{12} = m^1_{34} = 0$ which corresponds to a one-dimensional moduli space. Eqs. (29)–(33) (together with $\lambda = \mu = 0$) imply

$$\begin{aligned} T_2 = M^1_{13} = M^1_{14} = M^1_{23} = M^1_{24} &= 0 = M^0_{ij} \quad \forall i,j \ , \\ M^1_{12}\, M^1_{34} &= 4\Lambda^6 \ . \end{aligned} \tag{42}$$

[11] This excludes special relations among non-vanishing masses.

These constraints lead to the same singularity structure (cf. eqs. (13), (15)) as in the $SU(2)$ case discussed in section 2.

A different situation occurs for $m_{12}^0 = m_{34}^0 = m_{13}^0 = m_{24}^1 = m_{34}^1 = m = 0$, since eq. (33) is not automatically satisfied. The moduli space has dimension four and is constrained by

$$\begin{aligned} M_{14}^{0,1} = M_{23}^{0,1} = M_{24}^0 = M_{12}^1 = M_{13}^1 &= 0 \, , \\ T_2 \, M_{12}^0 \, M_{34}^0 &= 2\Lambda^6 \, , \\ M_{12}^0 \, M_{34}^1 - M_{13}^0 \, M_{24}^1 &= 0 \, . \end{aligned} \tag{43}$$

If we solve the last two equations for T_2 and M_{12}^0, say, we find for a Yukawa coupling of the form

$$Y(M) \sim T_2 + M_{12}^0 + M_{34}^0 + M_{13}^0 + M_{24}^1 + M_{34}^1 + \ldots \tag{44}$$

the singularities

$$Y(M) \sim \frac{2M_{34}^1 \Lambda^6}{M_{34}^0 M_{13}^0 M_{24}^1} + \frac{M_{13}^0 M_{24}^1}{M_{34}^1} + \ldots \, , \tag{45}$$

as M_{34}^0, M_{13}^0, M_{24}^1, or $M_{34}^1 \to 0$. There are four singular components but only three of them appear in the same term of the Yukawa couplings. In general, we will call l singular components *intersecting* if they give rise to a singularity of degree l in the Yukawa couplings when going to zero simultaneously. One finds that at most three intersecting singular components can be generated in the $Sp(4)$ model.

Before we turn to the discussion of the general situation in $Sp(2k)$, let us note that the $SU(2)$ gauge theory discussed in section 2 arises on the Higgs branch of the $Sp(4)$ gauge theory considered above. An expectation value of the antisymmetric tensor $A^{\alpha\beta}$ breaks $Sp(4) \to SU(2) \times SU(2)$, and one recovers two copies of the $SU(2)$ model of the previous section. This follows directly from ref. [31], where it was shown that the confining superpotential of the $Sp(2k)$ theory breaks into a sum of $SU(2)$-type superpotential terms when a VEV is given to the antisymmetric tensor. The D-flatness condition implies (up to gauge rotations)

$$\langle A^{\alpha\beta} \rangle = \begin{pmatrix} 0 & v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & -v \\ 0 & 0 & v & 0 \end{pmatrix} , \tag{46}$$

where $\frac{1}{2}v^2 = T_2 \equiv \frac{1}{8}\langle A_\alpha{}^\beta \rangle \langle A_\beta{}^\alpha \rangle$. Inserted into eqs. (22), one has

$$M_{ij}^0 = q_i^\alpha \epsilon_{\alpha\beta} q_j^\beta + q_i^{\alpha+2} \epsilon_{\alpha\beta} q_j^{\beta+2} \equiv \hat{M}_{ij}^{(1)} + \hat{M}_{ij}^{(2)} \, , \tag{47}$$

$$M_{ij}^1 = -v q_i^\alpha \epsilon_{\alpha\beta} q_j^\beta + v q_i^{\alpha+2} \epsilon_{\alpha\beta} q_j^{\beta+2} \equiv -v\hat{M}_{ij}^{(1)} + v\hat{M}_{ij}^{(2)} \, . \tag{48}$$

The two constraints (32) and (33) written in terms of $\hat{M}^{(1)}, \hat{M}^{(2)}$ read

$$v^2 \mathrm{Pf} \hat{M}^{(1)} + v^2 \mathrm{Pf} \hat{M}^{(2)} = 2\Lambda_{Sp(4)}^6 \, , \tag{49}$$

$$-v \mathrm{Pf} \hat{M}^{(1)} + v \mathrm{Pf} \hat{M}^{(2)} = 0 \, . \tag{50}$$

Using the scale matching condition [31] $\Lambda^6_{Sp(4)} = v^2 \Lambda^4_{SU(2)}$, we see that we have two copies of the $SU(2)$ moduli space with the correct constraints (8)

$$\mathrm{Pf}\hat{M}^{(1)} = \Lambda^4_{SU(2)} \ , \quad \mathrm{Pf}\hat{M}^{(2)} = \Lambda^4_{SU(2)} \ . \tag{51}$$

So far we focused on a non-perturbative gauge group $G_{\mathrm{NP}} = Sp(4)$ and saw that up to four singular components in the Yukawa couplings can be generated. More singular components arise in an $Sp(2k)$ gauge theory, once we have $k > 2$. The main point of the mechanism that generates singularities in the Yukawa couplings is the observation that a quantum constraint for a product of $l+1$ different moduli fields gives rise to l singular components when inserted in generic Yukawa couplings. We first restrict ourselves to the case where all singular components are intersecting in the sense defined above and where their number is equal to the dimension of the moduli space. As a consequence, only one of the k constraints on the quantum moduli space is relevant, namely the one of the form (25). The $k-1$ homogeneous constraints, for which the right hand side vanishes, must be trivially satisfied; else they would either involve non-singular moduli fields or produce additional singular components which do not intersect the others.

We are interested in the smallest non-perturbative $Sp(2k)$ gauge group that is able to generate l intersecting singular components in the Yukawa couplings. Because of (25) and (26) we need to look at a term of the form

$$T_2\, T_3 \ldots T_l\, \mathrm{Pf} M^0 \tag{52}$$

to find the minimal k required for a quantum constraint for $l+1$ different moduli fields. The condition (26) now reads $l + m + \sum r i_r = \sum_{r=2}^{l} r = \frac{1}{2} l(l+1) - 1 \ = \ 2(k-1)$, i.e., the considered term is allowed if $4k = l(l+1)+2$. As k is integer there is not a solution for each l, but whenever

$$4k \geq l(l+1) + 2 \ , \tag{53}$$

a term with at least $l+1$ different factors in the constraint (25) is possible. Inserted into generic Yukawa couplings this yields a singularity of degree l. Unfortunately, the coefficients of the sum (25) are not known for general k; there are terms which are allowed by the symmetries but are not generated by the gauge dynamics because their coefficients vanish. Therefore the condition (53) is only necessary but not sufficient for the appearance of the terms considered above. (Up to $k=4$ the coefficients are known [31], and the condition (53) is also sufficient.)

If the dimension of the moduli space is greater than the number of intersecting singular components, then the condition (53) to get l intersecting singular components is no longer valid. This is because the $k-1$ homogeneous constraints now play a role in that they can lead to more intersecting singular components, as we have seen in our last example of the $Sp(4)$ model. In this example we found $l=3$ intersecting singular components. The solutions of the constraint equations for $Sp(2k)$ get more and more complex with increasing k. The case $k=3$ is still tractable, and it turns out that one can have $l=5$.

4 Heterotic vacua with non-perturbative gauge group $U(2k)$

So far we discussed heterotic vacua in $d = 4$ which are related to heterotic $SO(32)$ vacua in $d = 6$ with small instantons sitting on smooth points of the $K3$. The situation is more involved when an $SO(32)$ instanton shrinks on a singularity of the underlying CFT or K3 manifold. (For simplicity we only discuss the case of an A_1 singularity.) In this case one has to distinguish between instantons 'with vector structure' and 'without vector structure' [5].[12] For instantons without vector structure the instanton moduli space is conjectured to be isomorphic to the Higgs branch of a $U(2k)$ gauge theory with 16 hypermultiplets in the fundamental ($\square$) $2k$-dimensional representation and two antisymmetric tensors ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) in the $k(2k-1)$-dimensional representation [5, 14]. Furthermore, the $U(1)$ factor is spontaneously broken by the Green-Schwarz mechanism and the the low energy effective theory only has $SU(2k)$ massless gauge boson [5].

Small instantons with vector structure on an A_1 singularity show a more complicated behaviour [6, 7, 8, 14, 34]. The case where less than four instantons coalesce on the singularity is not fully understood yet, while the moduli space of four (and more) instantons on the singularity has a Higgs branch and a Coulomb branch. On the Coulomb branch the dimension of the moduli space has been reduced by 29, but an additional tensor multiplet is present.

Here we restrict ourselves to k instantons without vector structure shrinking on an A_1 singularity of the $K3$ fibre. Again the computation of the spectrum in $d = 4$ can be performed following the methods of ref. [11]. One finds two flavours of fundamentals, $q_i, \bar{q}_i, i = 1, 2$, as well as one flavour of antisymmetric tensors $A, \bar{A}$.[13] As before, this is an asymptotically free gauge theory with $c_{SU(2k)} = 2k - 4 \cdot \frac{1}{2} - 2(k-1) = 0$ for any k. Consequently, this theory confines below $\Lambda_{SU(2k)}$, but no non-perturbative superpotential is generated by the strong coupling dynamics.

For simplicity, we first concentrate on $G_{\rm NP} = SU(4)$ where $A \cong \bar{A}$ holds. Therefore the two antisymmetric tensors A_r, $r = 1, 2$, transform as a doublet under an additional $SU(2)$ flavour symmetry. The low energy degrees of freedom are found to be [26]

$$
\begin{aligned}
M^0_{ij} &:= q_i \bar{q}_j = q_i^\alpha \bar{q}_{j\alpha} , \quad i, j = 1, 2 , \\
M^1_{ij} &:= q_i A^2 \bar{q}_j = \tfrac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{rs} q_i^\alpha A_r^{\beta\gamma} A_s^{\delta\lambda} \bar{q}_{j\lambda} , \\
H_r &:= q_i A_r q_j = \tfrac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{ij} q_i^\alpha A_r^{\beta\gamma} q_j^\delta , \quad r = 1, 2 , \\
\bar{H}_r &:= \bar{q}_i A_r \bar{q}_j = \tfrac{1}{2}\epsilon^{ij} \bar{q}_{i\alpha} A_r^{\alpha\beta} \bar{q}_{j\beta} ,
\end{aligned}
\tag{54}
$$

[12] This terminology refers to properties of the $SO(32)$ connection at infinity. See refs. [5, 6] for details.

[13] By giving an appropriate vacuum expectation value to one of the two antisymmetric tensors of $SU(2k)$ one arrives at the $Sp(2k)$ gauge theory with the precise spectrum discussed in section 3.

$$T_{rs} \ := \ A_r A_s \ = \ \tfrac{1}{8}\epsilon_{\alpha\beta\gamma\delta}A_r^{\alpha\beta}A_s^{\gamma\delta} \ .$$

These singlet fields satisfy the additional constraints

$$\begin{aligned} \det T \det M^0 - \epsilon^{rs}\epsilon^{tu}T_{rt}H_s\bar{H}_u - \det M^1 &= \Lambda^8, \\ \epsilon^{ij}\epsilon^{kl}M^0_{ik}M^1_{jl} + \epsilon^{rs}H_r\bar{H}_s &= 0 \ . \end{aligned} \tag{55}$$

Implementing these constraints in the superpotential via Lagrange multipliers λ and μ, we have

$$\begin{aligned} W \ = \ & m_{\hat{I}}(M,H,T)\,\hat{Q}^{\hat{I}}\hat{Q}^{\hat{I}} + Y_{IJK}(M,H,T)\,Q^IQ^JQ^K \\ & +\lambda\,(\det T \det M^0 - TH\bar{H} - \det M^1 - \Lambda^8) + \mu\,(M^0M^1 + H\bar{H}) \\ & +\sum_{i,j}\left(m^{(0)}_{ij}(M^0_{ij})^2 + m^{(1)}_{ij}(M^1_{ij})^2\right) + \sum_{r\le s} m^{(2)}_{rs}(T_{rs})^2 + \sum_r \left(m^{(3)}_r(H_r)^2 + m^{(4)}_r(\bar{H}_r)^2\right) \\ & + \dots \ . \end{aligned} \tag{56}$$

Using the equations of motion, it is easy to show that, in direct analogy to the $Sp(4)$ case,

$$\langle W\rangle = \sum_{r\le s} m^{(2)}_{rs}(T_{rs})^2 - \lambda\Lambda^8 \ . \tag{57}$$

As for $Sp(4)$, the superpotential vanishes at the minimum for $\lambda = 0$. However, in this case we were not able to show that this condition is also sufficient. Nevertheless we continue to consider only solutions with $\lambda = 0$. As before, $\lambda = 0$ implies $\mu = 0$ for generic mass terms, and the equations of motion can be easily solved. All fields with non-vanishing mass terms are set to zero, while the remaining degrees of freedom must satisfy the constraints (55).

The structure of the singularities again depends on the mass terms for the confined degrees of freedom. For example, mass terms for the fields $M^1_{ij}, H_r, \bar{H}_r, M^0_{12}, M^0_{21}$ and T_{12} $(i,j,r = 1,2)$ result in a three-dimensional moduli space described by

$$\begin{aligned} M^1_{ij} = H_r = \bar{H}_r = M^0_{12} = M^0_{21} = T_{12} &= 0 \ , \\ T_{11}T_{22}M^0_{11}M^0_{22} &= \Lambda^8 \ . \end{aligned} \tag{58}$$

Inserting this into generic Yukawa couplings results in a singular locus with three intersecting components:

$$Y(M) \sim \frac{\Lambda^8}{T_{22}M^0_{11}M^0_{22}} + \dots \ . \tag{59}$$

A more complicated structure of the moduli space is obtained if only M^1_{12}, M^1_{21}, M^1_{22}, M^0_{12}, M^0_{21}, H_1, $\bar{H}_2$ and T_{12} obtain mass terms. Solving the equations of motion, one finds a five-dimensional moduli space:

$$\begin{aligned} M^1_{12} = M^1_{21} = M^1_{22} = M^0_{12} = M^0_{21} = H_1 = \bar{H}_2 = T_{12} &= 0 \ , \\ T_{11}T_{22}M^0_{11}M^0_{22} &= \Lambda^8 \ , \\ M^0_{22}M^1_{11} - H_2\bar{H}_1 &= 0 \end{aligned} \tag{60}$$

If we eliminate T_{11} and M^0_{22} by these two constraints, a generic Yukawa coupling will have singularities of the form

$$Y(M) \sim \frac{M^1_{11}\Lambda^8}{T_{22}M^0_{11}H_2\bar{H}_1} + \frac{H_2\bar{H}_1}{M^1_{11}} + \dots \ . \tag{61}$$

There are five singular components four of which are intersecting in that they appear in the same coupling term.

In the general case of $G_{\rm NP} = SU(2k)$, $k > 2$, the situation is more complex. The antisymmetric tensor $A^{\alpha\beta}$ has now to be distinguished from its conjugate $\bar{A}_{\alpha\beta}$. Therefore the confined spectrum differs from the $SU(4)$ case. The low energy degrees of freedom are[14] [26]

$$\begin{aligned}
M^l_{ij} &:= q_i(\bar{A}A)^l\bar{q}_j \ = \ q^\alpha_i\bar{A}_{\alpha\beta_1}A^{\beta_1\beta_2}\cdots A^{\beta_{2l-1}\gamma}\bar{q}_{j\gamma}\ , \quad i,j=1,2,\ l=0,\dots,k-1\ ,\\
P^m &:= q(\bar{A}A)^m\bar{A}q \ = \ \epsilon^{ij}q^\alpha_i\bar{A}_{\alpha\beta_1}A^{\beta_1\beta_2}\cdots\bar{A}_{\beta_{2m}\gamma}q^\gamma_j\ , \quad m=0,\dots,k-2\ ,\\
\bar{P}^m &:= \bar{q}A(\bar{A}A)^m\bar{q} \ = \ \epsilon^{ij}\bar{q}_{i\alpha}A^{\alpha\beta_1}\bar{A}_{\beta_1\beta_2}\cdots A^{\beta_{2m}\gamma}q_{j\gamma}\ ,\\
B_0 &:= A^k \ = \ \epsilon_{\alpha_1\cdots\alpha_{2k}}A^{\alpha_1\alpha_2}\cdots A^{\alpha_{2k-1}\alpha_{2k}}\ ,\\
\bar{B}_0 &:= \bar{A}^k \ = \ \epsilon^{\alpha_1\cdots\alpha_{2k}}\bar{A}_{\alpha_1\alpha_2}\cdots\bar{A}_{\alpha_{2k-1}\alpha_{2k}}\ ,\\
B_2 &:= A^{k-1}qq \ = \ \epsilon_{\alpha_1\cdots\alpha_{2k}}A^{\alpha_1\alpha_2}\cdots A^{\alpha_{2k-3}\alpha_{2k-2}}q^{\alpha_{2k-1}}q^{\alpha_{2k}}\ ,\\
\bar{B}_2 &:= \bar{A}^{k-1}\bar{q}\bar{q} \ = \ \epsilon^{\alpha_1\cdots\alpha_{2k}}\bar{A}_{\alpha_1\alpha_2}\cdots\bar{A}_{\alpha_{2k-3}\alpha_{2k-2}}\bar{q}_{\alpha_{2k-1}}\bar{q}_{\alpha_{2k}}\ ,\\
T_n &:= (\bar{A}A)^n \ = \ \bar{A}_{\alpha\beta_1}A^{\beta_1\beta_2}\cdots A^{\beta_{2n-1}\alpha}\ , \quad n=1,\dots,k-1\ .
\end{aligned}\tag{62}$$

These fields are not all independent but obey constraint equations which, in general, are quite involved but can be derived from the confining superpotential of the model with three quark flavours ($N_f = 3$) [26]. By the gauge and global symmetries this superpotential is forced to be of the form

$$\begin{aligned}
W_{N_f=3} \ = \ \frac{1}{\Lambda^{4k-1}}\Bigg(&\sum_{\substack{\{i_r\},j,m,n,\\ l_1,l_2,l_3}} c_{ijlmn}\prod_{r=1}^{k-1}(T_r)^{i_r}(B_0\bar{B}_0)^j(M^{l_1}M^{l_2}M^{l_3}+\alpha_{mn}M^mP^n\bar{P}^n)\\
&+\beta M^0B_2\bar{B}_2\Bigg)\ ,
\end{aligned}\tag{63}$$

where flavour indices have been suppressed, and the sum goes over all integers

$$\begin{gathered}
0\le l_1,l_2,l_3,m\le k-1\ , \quad 0\le n\le k-2\ , \quad i_r\ge 0\ , \quad j=0,1\ ,\\
\text{with} \quad \left.\begin{matrix} l_1+l_2+l_3\\ m+2n+1\end{matrix}\right\} + kj + \sum_{r=1}^{k-1} r i_r = 2(k-1)\ .
\end{gathered}\tag{64}$$

[14]Note that the 'mesons' $M, P, \bar{P}$ are identical to the meson singlets of the dual magnetic theory found in ref. [35].

Again, the coefficients can be calculated by requiring that the equations of motion reproduce the classical constraints. The quantum constraints on the $N_f = 2$ moduli space are obtained by giving a large mass to one quark flavour and integrating out the massive modes. Only one of these constraints differs from the corresponding classical constraint. It has the form

$$\sum_{\substack{\{i_r\},j,\\ l,m,n}} \tilde{c}_{ijlmn} \prod_{r=1}^{k-1} (T_r)^{i_r} (B_0\bar{B}_0)^j (M^l M^m + \tilde{\alpha}_n P^n \bar{P}^n) \ + \ \tilde{\beta} B_2 \bar{B}_2 = \Lambda^{4k} \ . \tag{65}$$

where it is summed over all i_r, j, l, m, n that satisfy

$$\left.\begin{array}{c} l+m \\ 2n+1 \end{array}\right\} + kj + \sum_{r=1}^{k-1} r i_r = 2(k-1) \ . \tag{66}$$

To see how many intersecting singular components are possible for general k, we look again at products of $l+1$ moduli fields in eq. (65).[15] Relevant terms with $l+1$ factors which are allowed for minimal k are of the form

$$T_1 \cdots T_{l-1} M^0 M^0 \quad \text{or}$$
$$T_1 \cdots T_{l-3} B_0 \bar{B}_0 M^0 M^0 \ .$$

By eq. (66), the term appearing in the first line is allowed if $4k = l(l-1)+4$, while the second one requires $2k = (l-2)(l-3)+4$. One finds that for l intersecting singular components to be possible, necessarily

$$2k \geq (l-2)(l-3) + 4 \qquad \text{if} \qquad l \leq 6 \ , \tag{67}$$

$$4k \geq l(l-1) + 4 \qquad \text{if} \qquad l \geq 7 \ . \tag{68}$$

Let us summarize. In this paper we expanded on a mechanism suggested in ref. [11] which gives rise to singular couplings in the low energy effective theory of $N = 1$ heterotic string vacua. In addition to the perturbative gauge group of a string vacuum, also a confined non-perturbative gauge group can be present whose strong coupling dynamics renders the Yukawa and gauge couplings singular. This phenomenon occurs for asymptotically free gauge theories with the additional property that c, defined in eq. (6), vanishes. Such non-perturbative gauge theories do appear in $K3$ fibred Calabi–Yau compactifications and are directly related to non-perturbative gauge theories in $d = 6$. The structure of the singularities are qualitatively similar to the singularities in Calabi–Yau compactifications, but a more detailed quantitative comparison is still lacking. It would also be interesting to extend our work for other classes of heterotic vacua (see for example ref. [36]) as well as to study the possibility of extremal transitions between $N = 1$ heterotic string vacua along the lines of refs. [37, 20].

[15] As in the $Sp(2k)$ case, the following argument is restricted to the case where the dimension of the moduli space equals the number of intersecting singular components.

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